

LONG TIME BEHAVIOUR OF RANDOM WALKS ON THE INTEGER LATTICE

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ABSTRACT. We consider an irreducible finite range random walk on the d -dimensional integer lattice and study asymptotic behaviour of its transition function $p(n; x)$. In particular, for simple random walk our asymptotic formula is valid as long as $n(n - |x|_1)^{-2}$ tends to zero.

1. INTRODUCTION

In 1921, with the article [11] Pólya pioneered research on the simple random walk on the integer lattice. Using Fourier analysis he proved that $p(n; x)$, the n 'th step transition function, satisfies¹

$$\begin{aligned} \lim_{n \rightarrow +\infty} (2n)^{\frac{d}{2}} p(2n; x) &= 2d^{\frac{d}{2}} (2\pi)^{-\frac{d}{2}}, \quad \text{if } |x|_1 \equiv 0 \pmod{2}, \\ \lim_{n \rightarrow +\infty} (2n-1)^{\frac{d}{2}} p(2n-1; x) &= 2d^{\frac{d}{2}} (2\pi)^{-\frac{d}{2}}, \quad \text{if } |x|_1 \equiv 1 \pmod{2}, \end{aligned}$$

for any $x \in \mathbb{Z}^d$. Essentially, Pólya's proof shows that (see Spitzer [13, Remark after P7.9])

$$p(n; x) = \begin{cases} 2d^{\frac{d}{2}} (2\pi n)^{-\frac{d}{2}} e^{-\frac{d}{2n}|x|_2^2} + o(n^{-\frac{d}{2}}) & \text{if } |x|_1 \equiv n \pmod{2}, \\ 0 & \text{otherwise,} \end{cases}$$

uniformly with respect to $n \in \mathbb{N}$ and $x \in \mathbb{Z}^d$. As it may be easily seen the local limit theorem is very inaccurate if $|x|_2$ is larger than \sqrt{n} . Further development of the Fourier method allowed to gain better control over the error term for large $|x|_2$ (see Smith [12], Spitzer [13, P7.10], Ney and Spitzer [10, Theorem 2.1]). Namely,

$$p(n; x) = \begin{cases} 2d^{\frac{d}{2}} (2\pi n)^{-\frac{d}{2}} e^{-\frac{d}{2n}|x|_2^2} + o(n^{-\frac{d}{2}+1}|x|_2^{-2}), & \text{if } |x|_1 \equiv n \pmod{2}, \\ 0 & \text{otherwise,} \end{cases}$$

uniformly with respect to $x \in \mathbb{Z}^d \setminus \{0\}$. Let us observe that the error in the approximation of $p(n; x)$ is additive and may become big compared to the first term. In many applications, it is desired to have an asymptotic formula for $p(n; x)$ valid on the largest possible region with respect to n and x . There are some results in this direction available. In particular, (see Lawler [8, Proposition 1.2.5], Lawler

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¹For $x \in \mathbb{R}^d$ and $p \in (1, \infty)$ we set $|x|_p = (|x_1|^p + \dots + |x_d|^p)^{1/p}$

and Limič [9, Theorem 2.3.11]) there is $\rho > 0$ such that for all $n \in \mathbb{N}$ and $x \in \mathbb{Z}^d$, if $|x|_2 \leq \rho n$ then

$$p(n; x) = \begin{cases} d^{\frac{d}{2}} (2\pi n)^{-\frac{d}{2}} e^{-\frac{d}{2n}|x|_2^2} (2 + \mathcal{O}(n^{-1}) + \mathcal{O}(n^{-3}|x|_2^4)), & \text{if } |x|_1 \equiv n \pmod{2}, \\ 0 & \text{otherwise.} \end{cases}$$

We want to emphasize that the above asymptotic formula is useful only in the region where $|x|_2 = o(n^{3/4})$. Therefore, there arises a natural question:

Is there an asymptotic formula for $p(n; x)$ which is valid on a larger region than $|x|_2 = o(n^{3/4})$?

The subject of the present article is to give a positive answer to the posed question. Although, the asymptotic will be formulated for the simple random walk, the actual result is valid for any irreducible finite range random walk with the mean zero or not (see Theorem 3.1). Before we state the main theorem, let us introduce some notation. For $\delta \in \mathcal{M} = \{x \in \mathbb{R}^d : |x|_1 < 1\}$ we set

$$(1) \quad \phi(\delta) = \max \{ \langle x, \delta \rangle - \log \kappa(x) : x \in \mathbb{R}^d \}$$

where κ is a function on \mathbb{R}^d defined by

$$\kappa(x) = \frac{1}{d} (\cosh x_1 + \cdots + \cosh x_d).$$

In Section 3 we prove the following theorem.

Theorem A. *For all $x \in \mathbb{Z}^d$ and $n \in \mathbb{N}$, if $|x|_1 \equiv n \pmod{2}$ then*

$$(2) \quad p(n; x) = (2\pi n)^{-\frac{d}{2}} (\det B_s)^{-\frac{1}{2}} e^{-n\phi(\delta)} (2 + \mathcal{O}(n^{-1}(1 - |\delta|_1)^{-2}))$$

otherwise, $p(n; x) = 0$, where $\delta = \frac{x}{n}$ and $s = \nabla \phi(\delta)$.

Some comments are in order. First, observe that the asymptotic formula (2) is valid in a region excluding only the case when $n(1 - |\delta|_1)^2$ stays bounded. Although, the function ϕ is positive convex and comparable to $|\cdot|_2^2$, it cannot be replaced in the asymptotic formula by $|\cdot|_2^2$ without introducing an additional error term, see Remark 1. For processes with continuous time it was observed by Davis in [2] that in order to get an upper bound for the heat kernel on a larger region one has to introduce a non-Gaussian factor. Therefore, Theorem A may be considered as a discrete time counterpart of [2]. Finally, however the quadratic form B_x is given explicitly by (7), the mapping $\mathcal{M} \ni \delta \mapsto s(\delta)$ is an implicit function. We want to stress the fact that when $|\delta|_1$ approaches one, the value of $|s|_2$ tends to infinity. In particular, the quadratic form B_s degenerates. For this reason a more convenient form of Theorem A is given in Corollary 3.2. Namely, for all $\epsilon > 0$, $x \in \mathbb{Z}^d$ and $n \in \mathbb{N}$, if $|x|_1 \leq n(1 - \epsilon)$ then

$$p(n; x) = \begin{cases} d^{\frac{d}{2}} (2\pi n)^{-\frac{d}{2}} e^{-n\phi(\delta)} (2 + \mathcal{O}(|\delta|_1) + \mathcal{O}(n^{-1})), & \text{if } |x|_1 \equiv n \pmod{2}, \\ 0 & \text{otherwise.} \end{cases}$$

The last asymptotic formula is useful as long as $|x|_1 = o(n)$.

Let us comment about the method of the proof. First, with a help of the Fourier inversion formula, we write $p(n; x)$ as an oscillatory integral. We split the integral

into two parts. The first part we analyse by the Laplace method. This is not a straightforward application of it, since the phase function degenerates as $|\delta|_1$ approaches one. To estimate the second part we develop a geometric argument, which allows us to control the way how the quadratic form B_s degenerates.

The result obtained in this article has already found an application in the study of subordinated random walks (see [1]) which are spread over all \mathbb{Z}^d and do not have second moment. Also the geometric method developed here can be successfully applied in much wider context. Namely, to study finitely supported isotropic random walks on affine buildings (see [14]). There is also an ongoing project to get the precise asymptotic formula for random walks with internal degrees of freedom extending the one obtained by Krámlí and Szász [7] (see also Guivarc'h [3]). Finally, Appendix A contains application of Theorem 3.1 to triangular and hexagonal lattices. This has to be compared with results recently obtained in [6, 5, 4].

1.1. Notation. We use the convention that C stands for a generic positive constant whose value can change from line to line. The set of positive integers is denoted by \mathbb{N} . Let $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$.

2. PRELIMINARIES

2.1. Random walks. Let $p(\cdot, \cdot)$ denote the transition density of a random walk on the d -dimensional integer lattice. Let $p(x) = p(0, x)$. For $n \in \mathbb{N}_0$ and $x \in \mathbb{Z}^d$ we set

$$p(n+1; x) = \sum_{y \in \mathbb{Z}^d} p(n; y)p(x-y)$$

and $p(1; x) = p(x)$. The support of p is denoted by \mathcal{V} , i.e.

$$\mathcal{V} = \{v \in \mathbb{Z}^d : p(v) > 0\}.$$

Let $\kappa : \mathbb{C}^d \rightarrow \mathbb{C}$ be an exponential polynomial defined by

$$\kappa(z) = \sum_{v \in \mathcal{V}} p(v)e^{\langle z, v \rangle},$$

where $\langle \cdot, \cdot \rangle$ is the standard scalar product on \mathbb{C}^d

$$\langle z, w \rangle = \sum_{j=1}^d z_j \overline{w_j}.$$

In particular, $\mathbb{R}^d \ni \theta \mapsto \kappa(i\theta)$ is the characteristic function of p . We set

$$(3) \quad \mathcal{U} = \{\theta \in [-\pi, \pi)^d : |\kappa(\theta)| = 1\}.$$

Finally, the interior of the convex hull of \mathcal{V} in \mathbb{R}^d is denoted by \mathcal{M} .

In this article, we study the asymptotic behaviour of transition functions of irreducible random walks. Let us recall that the random walk is *irreducible* if for each $x \in \mathbb{Z}^d$ there is $n \in \mathbb{N}$ such that $p(n; x) > 0$. By $d \in \mathbb{N}$ we denote the period of p , that is

$$d = \gcd \{n \in \mathbb{N} : p(n; 0) > 0\}.$$

Then the space \mathbb{Z}^d decomposes into r disjoint classes

$$X_j = \{x \in \mathbb{Z}^d : p(j + kr; x) > 0 \text{ for some } k \geq 0\}$$

for $j = 0, \dots, r-1$. We observe that for $j \in \{0, \dots, r-1\}$ and $x \in X_j$, if $n \not\equiv j \pmod{r}$ then

$$p(n; x) = 0.$$

For each $x \in \mathbb{Z}^d$, by m_x we denote the smallest $m \in \mathbb{N}$ such that $p(m; x) > 0$, thus $x/m_x \in \overline{\mathcal{M}}$. Notice that there is $C \geq 1$ such that for all $x \in \mathbb{Z}^d$

$$(4) \quad C^{-1}|x|_1 \leq m_x \leq C|x|_1.$$

Indeed, let $\{e_1, \dots, e_d\}$ be the standard basis of \mathbb{R}^d . Since

$$e_j = \sum_{v \in \mathcal{V}} m_{j,v} v, \quad \text{and} \quad -e_j = \sum_{v \in \mathcal{V}} m_{-j,v} v,$$

for some $m_{j,v}, m_{-j,v} \in \mathbb{N}_0$ satisfying

$$m_{e_j} = \sum_{v \in \mathcal{V}} m_{j,v}, \quad \text{and} \quad m_{-e_j} = \sum_{v \in \mathcal{V}} m_{-j,v},$$

by setting $\varepsilon_j = \text{sign} \langle x, e_j \rangle$ we get

$$x = \sum_{v \in \mathcal{V}} \left(\sum_{j=1}^d m_{\varepsilon_j j, v} |\langle x, e_j \rangle| \right) v.$$

Hence,

$$m_x \leq |x|_1 \sum_{j=1}^d (m_{e_j} + m_{-e_j}).$$

which, together with boundedness of $\overline{\mathcal{M}}$, implies (4).

Next, we observe that there is $K > 0$ such that for all $k \geq K$

$$p(kr; 0) > 0,$$

thus for all $x \in \mathbb{Z}^d$ and $n \geq Kr + m_x$

$$(5) \quad p(n; x) = \begin{cases} > 0 & \text{if } n \equiv m_x \pmod{r}, \\ = 0 & \text{otherwise.} \end{cases}$$

Since

$$(6) \quad \left\{ \frac{e_1}{m_{e_1}}, -\frac{e_1}{m_{-e_1}}, \dots, \frac{e_d}{m_{e_d}}, -\frac{e_d}{m_{-e_d}} \right\}$$

do not lay on the same affine hyperplane, the interior of the convex hull of (6) is a non-empty subset of \mathcal{M} .

For each $x \in \mathbb{R}^d$, by B_x we denote a quadratic form on \mathbb{R}^d defined by

$$(7) \quad B_x(u, u) = D_u^2 \log \kappa(x),$$

where D_u denotes the derivative along a vector u , i.e.

$$D_u f(x) = \left. \frac{d}{dt} f(x + tu) \right|_{t=0}.$$

Since

$$D_u \log \kappa(x) = \sum_{v \in \mathcal{V}} \frac{p(v)e^{\langle x, v \rangle}}{\kappa(x)} \langle u, v \rangle$$

and

$$D_u \left(\frac{p(v)e^{\langle x, v \rangle}}{\kappa(x)} \right) = \frac{p(v)e^{\langle x, v \rangle}}{\kappa(x)} \langle u, v \rangle - \sum_{v' \in \mathcal{V}} \frac{p(v)e^{\langle x, v \rangle}}{\kappa(x)} \cdot \frac{p(v')e^{\langle x, v' \rangle}}{\kappa(x)} \langle u, v' \rangle,$$

we may write

$$(8) \quad B_x(u, u) = \frac{1}{2} \sum_{v, v'} \frac{p(v)e^{\langle x, v \rangle}}{\kappa(x)} \cdot \frac{p(v')e^{\langle x, v' \rangle}}{\kappa(x)} \langle u, v - v' \rangle^2.$$

In particular, if the random walk is irreducible then the quadratic form B_x is positive definite.

Example 1. Let p be the transition function of the simple random walk on \mathbb{Z}^d , i.e.

$$p(e_j) = p(-e_j) = \frac{1}{2d}, \quad \text{for } j = 1, \dots, d.$$

Thus

$$\mathcal{V} = \{\pm e_j : j = 1, \dots, d\}, \quad \text{and} \quad \mathcal{M} = \{x \in \mathbb{R}^d : |x|_1 < 1\}.$$

Since

$$\kappa(z) = \frac{1}{2d} \sum_{j=1}^d (e^{z_j} + e^{-z_j}),$$

we get $\mathcal{U} = \{0, (-\pi, -\pi, \dots, -\pi)\}$. By a straightforward calculation we may find the quadratic form B_0 ,

$$B_0(u, u) = \frac{1}{(2d)^2} \sum_{j=1}^d \sum_{j'=1}^d (u_j + u_{j'})^2 + (u_j - u_{j'})^2 = \frac{1}{d} \langle u, u \rangle.$$

2.2. Function s . For the sake of completeness we provide the proof of the following well-known theorem.

Theorem 2.1. *For every $\delta \in \mathcal{M}$ a function $f(\delta, \cdot) : \mathbb{R}^d \rightarrow \mathbb{R}$ defined by*

$$f(\delta, x) = \langle x, \delta \rangle - \log \kappa(x)$$

attains its maximum at the unique point $s \in \mathbb{R}^d$ satisfying $\nabla \log \kappa(s) = \delta$.

Proof. Without loss of generality, we may assume $\nabla \log \kappa(0) = 0$. Indeed, otherwise we will consider

$$\tilde{\kappa}(z) = e^{-\langle z, v_0 \rangle} \kappa(z) = \sum_{v \in \tilde{\mathcal{V}}} p(v + v_0) e^{\langle z, v \rangle}$$

where $v_0 = \nabla \log \kappa(0)$ and $\tilde{\mathcal{V}} = \mathcal{V} - v_0$. Then $\tilde{\mathcal{M}}$, the interior of the convex hull of $\tilde{\mathcal{V}}$, is equal to $\mathcal{M} - v_0$. For $\tilde{\delta} = \delta - v_0$ we have

$$\tilde{f}(\tilde{\delta}, x) = \langle x, \delta - v_0 \rangle - \log \tilde{\kappa}(x) = \langle x, \delta \rangle - \log \kappa(x) = f(\delta, x).$$

We conclude that if s is the unique maximum of $\mathbb{R}^d \ni x \mapsto \tilde{f}(\tilde{\delta}, x)$, then it is also the unique maximum of $\mathbb{R}^d \ni x \mapsto f(\delta, x)$. Because

$$\nabla \log \tilde{\kappa}(x) = \nabla \log \kappa(x) - v_0$$

we get $\nabla \log \kappa(s) = \tilde{\delta} + v_0 = \delta$, proving the claim.

Fix $\delta \in \mathcal{M}$. Since $\nabla \kappa(0) = 0$, by Taylor's theorem we have

$$f(\delta, x) = \langle x, \delta \rangle + \mathcal{O}(|x|_2^2)$$

as $|x|_2$ approaches zero. Moreover, for any $x, u \in \mathbb{R}^d$

$$D_u^2 f(\delta, x) = -B_x(u, u),$$

thus the function $\mathbb{R}^d \ni x \mapsto f(\delta, x)$ is strictly concave.

Let us observe that

$$0 = \nabla \kappa(0) = \sum_{v \in \mathcal{V}} p(v) \cdot v \in \overline{\mathcal{M}}.$$

Since \mathcal{M} is not empty, the set \mathcal{V} cannot be contained in an affine hyperplane, thus, $0 \in \mathcal{M}$.

Now, $\delta \in \mathcal{M}$ implies that there are $v_1, \dots, v_d \in \partial \mathcal{M} \cap \mathcal{V}$ such that δ belongs to the convex hull of $\{0, v_1, \dots, v_d\}$, i.e. there are $t_0, t_1, \dots, t_d \in [0, 1]$ satisfying

$$\delta = t_0 \cdot 0 + \sum_{j=1}^d t_j \cdot v_j = \sum_{j=1}^d t_j \cdot v_j,$$

Because $\delta \notin \partial \mathcal{M}$ we must have $t_0 > 0$, thus $\sum_{j=1}^d t_j < 1$. Hence,

$$\sum_{j=1}^d t_j \log \kappa(x) \geq \sum_{j=1}^d t_j (\log p(v_j) + \langle x, v_j \rangle) = \sum_{j=1}^d t_j \log p(v_j) + \langle x, \delta \rangle,$$

and we get

$$f(\delta, x) = \langle x, \delta \rangle - \log \kappa(x) \leq \left(\sum_{j=1}^d t_j - 1 \right) \log \kappa(x) - \sum_{j=1}^d t_j \log p(v_j),$$

which implies that

$$\lim_{|x|_2 \rightarrow \infty} f(\delta, x) = -\infty,$$

because

$$\lim_{|x|_2 \rightarrow \infty} \log \kappa(x) = +\infty,$$

and the proof is finished. \square

In the rest of the article, given $\delta \in \mathcal{M}$ by s we denote the unique solution to

$$(9) \quad \delta = \nabla \log \kappa(s) = \sum_{v \in \mathcal{V}} \frac{p(v) e^{\langle s, v \rangle}}{\kappa(s)} \cdot v.$$

Let $\phi : \mathcal{M} \rightarrow \mathbb{R}$ be defined by

$$(10) \quad \phi(\delta) = \max\{\langle x, \delta \rangle - \log \kappa(x) : x \in \mathbb{R}^d\},$$

thus, by Theorem 2.1,

$$(11) \quad \phi(\delta) = \langle \delta, s \rangle - \log \kappa(s).$$

By (9), for any $u \in \mathbb{R}^d$,

$$\langle \delta, u \rangle = D_u \log \kappa(s).$$

Hence, for $u, u' \in \mathbb{R}^d$

$$\langle u, u' \rangle = D_u(D_{u'} \log \kappa(s)) = \sum_{j=1}^d D_j D_{u'} \log \kappa(s) D_u s_j = B_s(D_u s, u'),$$

i.e, $D_u s = B_s^{-1} u$. Therefore, we can calculate

$$\nabla \phi(\delta) = s + \sum_{j=1}^d \delta_j \nabla s_j - \sum_{j=1}^d D_j \log \kappa(s) \nabla s_j = s,$$

thus,

$$D_u^2 \phi(\delta) = D_u(\langle u, s \rangle) = B_s^{-1}(u, u).$$

In particular, ϕ is a convex function on \mathcal{M} . Let $\delta_0 = \nabla \log \kappa(0)$. By Taylor's theorem, we have

$$(12) \quad \phi(\delta) = \frac{1}{2} B_0^{-1}(\delta - \delta_0, \delta - \delta_0) + \mathcal{O}(|\delta - \delta_0|_2^3)$$

as δ approaches δ_0 . We claim that

Claim 1. For all $\delta \in \mathcal{M}$ ²

$$\phi(\delta) \asymp B_0^{-1}(\delta - \delta_0, \delta - \delta_0).$$

Since ϕ is convex and satisfies (12), it is enough to show that ϕ is bounded from above. Given $\delta \in \mathcal{M}$, let $v_0 \in \mathcal{V}$ be any vector satisfying

$$\langle s, v_0 \rangle = \max \{ \langle s, v \rangle : v \in \mathcal{V} \}.$$

Because

$$\langle s, \delta \rangle - \langle s, v_0 \rangle = \sum_{v \in \mathcal{V}} \frac{p(v) e^{\langle s, v \rangle}}{\kappa(s)} \langle s, v - v_0 \rangle \leq 0,$$

we get

$$\begin{aligned} \phi(\delta) &= \langle s, \delta \rangle - \log \kappa(s) \leq \langle s, \delta \rangle - \log(p(v_0) e^{\langle s, v_0 \rangle}) \\ &\leq -\log p(v_0), \end{aligned}$$

proving the claim.

Example 2. Let p be the transition density of the simple random walk on \mathbb{Z} . Then $\mathcal{V} = \{-1, 1\}$, $\mathcal{U} = \{0, -\pi\}$ and $\mathcal{M} = (-1, 1)$. For $\delta \in \mathcal{M}$, we have

$$e^s = \sqrt{\frac{1+\delta}{1-\delta}},$$

² $A \asymp B$ means that $cB \leq A \leq CB$, for some constants $c, C > 0$.

and

$$\kappa(s) = \frac{e^s + e^{-s}}{2} = \frac{1}{\sqrt{1 - \delta^2}}.$$

Hence, using (11) we obtain

$$\phi(\delta) = \frac{1}{2}(1 - \delta) \log(1 - \delta) + \frac{1}{2}(1 + \delta) \log(1 + \delta).$$

In general, there is no explicit formula for the function ϕ . By implicit function theorem, the function s is real analytic on \mathcal{M} . In particular, s is bounded on any compact subset of \mathcal{M} . From the other side, $|s|_2$ approaches infinity when δ tends to $\partial\mathcal{M}$. To see this, denote by \mathcal{F} a facet of \mathcal{M} such that δ approaches $\partial\mathcal{M} \cap \mathcal{F}$. Let u be an outward unit normal vector to \mathcal{M} at \mathcal{F} . Then for each $v_1 \in \mathcal{F} \cap \mathcal{V}$ and $v_2 \in \mathcal{V} \setminus \mathcal{F}$ we have

$$\begin{aligned} \langle v_1 - \delta, u \rangle &= \sum_{v \in \mathcal{V}} \frac{p(v)e^{\langle s, v \rangle}}{\kappa(s)} \langle v_1 - v, u \rangle \\ &= \sum_{v \in \mathcal{V} \setminus \mathcal{F}} \frac{p(v)e^{\langle s, v \rangle}}{\kappa(s)} \langle v_1 - v, u \rangle \geq \frac{p(v_2)e^{\langle s, v_2 \rangle}}{\kappa(s)} \langle v_1 - v_2, u \rangle. \end{aligned}$$

Therefore, for any $v \in \mathcal{V} \setminus \mathcal{F}$

$$(13) \quad \lim_{\delta \rightarrow \partial\mathcal{M} \cap \mathcal{F}} \frac{e^{\langle s, v \rangle}}{\kappa(s)} = 0.$$

The next theorem provides a control over the speed of convergence in (13).

Theorem 2.2. *There are constants $\eta \geq 1$ and $C > 0$ such that for all $\delta \in \mathcal{M}$, and $v \in \mathcal{V}$ we have*

$$\frac{e^{\langle s, v \rangle}}{\kappa(s)} \geq C \operatorname{dist}(\delta, \partial\mathcal{M})^\eta$$

where $s = s(\delta)$ satisfies $\delta = \nabla \log \kappa(s)$.

Proof. We consider any enumeration of elements of $\mathcal{V} = \{v_1, \dots, v_N\}$. Define

$$\Omega = \{\omega \in \mathcal{S} : \langle \omega, v_i \rangle \geq \langle \omega, v_{i+1} \rangle \text{ for } i = 1, \dots, N-1\}$$

where \mathcal{S} is the unit sphere in \mathbb{R}^d centred at the origin. Suppose $\Omega \neq \emptyset$ and let k be the smallest index such that points $\{v_1, \dots, v_k\}$ do not lay on the same facet of \mathcal{M} . Let us recall that a set \mathcal{F} is a facet of \mathcal{M} if there is $\lambda \in \mathcal{S}$ and $c \in \mathbb{R}$ such that for all $v \in \mathcal{V}$, $\langle \lambda, v \rangle \leq c$, and

$$\mathcal{F} = \operatorname{conv}\{v \in \mathcal{V} : \langle \lambda, v \rangle = c\}.$$

Since $\{v_1, \dots, v_k\}$ do not lay on the same facet of \mathcal{M} and Ω is a compact set, there is $\epsilon > 0$ such that for all $\omega \in \Omega$ we have

$$(14) \quad \langle \omega, v_1 \rangle \geq \langle \omega, v_k \rangle + \epsilon.$$

Let \mathcal{F} be a facet containing $\{v_1, \dots, v_{k-1}\}$. For $\frac{x}{|x|_2} \in \Omega$ and

$$\delta = \sum_{v \in \mathcal{V}} \frac{p(v)e^{\langle x, v \rangle}}{\kappa(x)} \cdot v,$$

we have

$$\begin{aligned} \text{dist}(\delta, \partial\mathcal{M}) &\leq \langle \lambda, v_1 - \delta \rangle = \sum_{v \in \mathcal{V} \setminus \mathcal{F}} \frac{p(v)e^{\langle x, v \rangle}}{\kappa(x)} \langle \lambda, v_1 - v \rangle \\ &\leq C \frac{e^{\langle x, v_k \rangle}}{\kappa(x)}. \end{aligned}$$

Since

$$p(v_1)e^{\langle x, v_1 \rangle} \leq \kappa(x) \leq e^{\langle x, v_1 \rangle}$$

we obtain

$$\text{dist}(\delta, \partial\mathcal{M}) \leq Ce^{\langle x, v_k - v_1 \rangle}.$$

In particular, for $1 \leq j \leq k$, we have

$$\frac{e^{\langle x, v_j \rangle}}{\kappa(x)} \geq C \text{dist}(\delta, \partial\mathcal{M}).$$

If $j > k$, we can estimate

$$\begin{aligned} \frac{e^{\langle x, v_j \rangle}}{\kappa(x)} &\geq e^{\langle x, v_j - v_1 \rangle} = \left(e^{\langle x, v_k - v_1 \rangle} \right)^{\langle x, v_1 - v_j \rangle / \langle x, v_1 - v_k \rangle} \\ &\geq C \text{dist}(\delta, \partial\mathcal{M})^{\langle x, v_1 - v_j \rangle / \langle x, v_1 - v_k \rangle} \end{aligned}$$

what finishes the proof since by (14)

$$1 \leq \frac{\langle x, v_1 - v_j \rangle}{\langle x, v_1 - v_k \rangle} \leq \epsilon^{-1} |v_1 - v_j|_2. \quad \square$$

2.3. Analytic lemmas. For a multi-index $\sigma \in \mathbb{N}^d$ we denote by X_σ a multi-set containing $\sigma(i)$ copies of i . Let Π_σ be a set of all partitions of X_σ . For the convenience of the reader we recall the following lemma.

Lemma 2.3 (Faà di Bruno's formula). *There are positive constants c_π , $\pi \in \Pi_\sigma$, such that for sufficiently smooth functions $f : S \rightarrow T$, $F : T \rightarrow \mathbb{R}$, $T \subset \mathbb{R}$, $S \subset \mathbb{R}^d$, we have*

$$\partial^\sigma F(f(s)) = \sum_{\pi \in \Pi_\sigma} c_\pi \frac{d^m}{dt^m} \Big|_{t=f(s)} F(t) \prod_{j=1}^m \partial^{B_j} f(s)$$

where $\pi = \{B_1, \dots, B_m\}$.

For a multi-set B containing $\sigma(i)$ copies of i we set $B! = \sigma!$. Let us observe that for

$$F(t) = \frac{1}{2-t}, \quad \text{and} \quad f(s) = \prod_{j=1}^d \frac{1}{1-s_j},$$

the function $F(f(s))$ is real-analytic in some neighbourhood of $s = 0$. Hence, there is $C > 0$ such that for every $\sigma \in \mathbb{N}^d$

$$(15) \quad \partial^\sigma F(f(0)) = \sum_{\pi \in \Pi_\sigma} c_\pi m! \prod_{j=1}^m B_j! \leq C^{|\sigma|+1} \sigma!.$$

Using Lemma 2.3 one can show

Lemma 2.4. *Let $\mathcal{V} \subset \mathbb{R}^d$ be a set of finite cardinality. Assume that for each $v \in \mathcal{V}$, we are given $a_v \in \mathbb{C}$, and $b_v > 0$. Then for $z = x + i\theta \in \mathbb{C}^d$ such that*

$$|\theta|_2 \leq (2 \cdot \max\{|v|_2 : v \in \mathcal{V}\})^{-1},$$

we have

$$(16) \quad \left| \sum_{v \in \mathcal{V}} b_v e^{\langle z, v \rangle} \right| \geq \frac{1}{\sqrt{2}} \sum_{v \in \mathcal{V}} b_v e^{\langle x, v \rangle}.$$

Moreover, there is $C > 0$ such that for all $\sigma \in \mathbb{N}^d$

$$(17) \quad \left| \partial^\sigma \left\{ \frac{\sum_{v \in \mathcal{V}} a_v e^{\langle z, v \rangle}}{\sum_{v \in \mathcal{V}} b_v e^{\langle z, v \rangle}} \right\} \right| \leq C^{|\sigma|} \sigma! \frac{\sum_{v \in \mathcal{V}} |a_v| e^{\langle x, v \rangle}}{\sum_{v \in \mathcal{V}} b_v e^{\langle x, v \rangle}}.$$

Proof. We start by proving (16). We have

$$\begin{aligned} \left| \sum_{v \in \mathcal{V}} b_v e^{\langle z, v \rangle} \right|^2 &= \sum_{v, v' \in \mathcal{V}} b_v b_{v'} e^{\langle x, v+v' \rangle} \cos \langle \theta, v - v' \rangle \\ &\geq \sum_{v, v' \in \mathcal{V}} b_v b_{v'} e^{\langle x, v+v' \rangle} \left(1 - \frac{\langle \theta, v - v' \rangle^2}{2} \right) \\ &\geq \frac{1}{2} \left(\sum_{v \in \mathcal{V}} b_v e^{\langle x, v \rangle} \right)^2 \end{aligned}$$

because $|\langle \theta, v - v' \rangle| \leq 1$.

For the proof of (17), it is enough to show

$$(18) \quad \left| \partial^\sigma \left\{ \frac{1}{\sum_{v \in \mathcal{V}} b_v e^{\langle z, v \rangle}} \right\} \right| \leq C^{|\sigma|+1} \sigma! \frac{1}{\sum_{v \in \mathcal{V}} b_v e^{\langle x, v \rangle}}.$$

Indeed, since

$$(19) \quad \left| \partial^\alpha \left\{ \sum_{v \in \mathcal{V}} a_v e^{\langle z, v \rangle} \right\} \right| \leq \sum_{v \in \mathcal{V}} |a_v| \cdot |v^\alpha| e^{\langle x, v \rangle} \leq C^{|\alpha|} \sum_{v \in \mathcal{V}} |a_v| e^{\langle x, v \rangle},$$

by (18) and the Leibniz's rule we obtain (17). To show (18), we use Faà di Bruno's formula with $F(t) = 1/t$. By Lemma 2.3 together with estimates (16) and (19) we get

$$\begin{aligned} \left| \partial^\sigma \left\{ \frac{1}{\sum_{v \in \mathcal{V}} b_v e^{\langle z, v \rangle}} \right\} \right| &\leq \sum_{\pi \in \Pi_\sigma} c_\pi m! \left(\sum_{v \in \mathcal{V}} b_v e^{\langle x, v \rangle} \right)^{-m-1} \prod_{j=1}^m \left| \partial^{B_j} \left\{ \sum_{v \in \mathcal{V}} b_v e^{\langle z, v \rangle} \right\} \right| \\ &\leq C^{|\sigma|} \frac{1}{\sum_{v \in \mathcal{V}} b_v e^{\langle x, v \rangle}} \sum_{\pi \in \Pi_\sigma} c_\pi m! \\ &\leq C^{|\sigma|+1} \frac{1}{\sum_{v \in \mathcal{V}} b_v e^{\langle x, v \rangle}}, \end{aligned}$$

where in the last inequality we have used (15). □

3. HEAT KERNELS

In this section we show the asymptotic behaviour of the n 'th step transition density of an irreducible random walk on the integer lattice \mathbb{Z}^d . Before we state and proof the main theorem, let us present the following example.

Example 3. Let p be the transition function of the simple random walk on \mathbb{Z} . If $x \equiv n \pmod{2}$ then

$$p(n; x) = \frac{1}{2^n} \frac{n!}{\left(\frac{n-x}{2}\right)! \left(\frac{n+x}{2}\right)!}.$$

Let us recall Stirling's formula

$$n! = \sqrt{2\pi n} n^{n+\frac{1}{2}} e^{-n} (1 + \mathcal{O}(n^{-1})).$$

Hence, we have

$$\begin{aligned} p(n; x) &= \frac{1}{\sqrt{2\pi}} \frac{n^{n+\frac{1}{2}}}{(n-x)^{\frac{n-x+1}{2}} (n+x)^{\frac{n+x+1}{2}}} (1 + \mathcal{O}((n-x)^{-1}) + \mathcal{O}((n+x)^{-1})) \\ &= \frac{1}{\sqrt{2\pi n}} (1 - \delta^2)^{-\frac{1}{2}} e^{-n\phi(\delta)} (1 + \mathcal{O}(n^{-1} \text{dist}(\delta, \{-1, 1\})^{-1})) \end{aligned}$$

where $\delta = \frac{x}{n}$ and $\phi(\delta) = \frac{1}{2}(1 - \delta) \log(1 - \delta) + \frac{1}{2}(1 + \delta) \log(1 + \delta)$.

Theorem 3.1. *Let p be an irreducible random walk on \mathbb{Z}^d . Let r be its period and X_0, \dots, X_{r-1} the partition of \mathbb{Z}^d into aperiodic classes. There is $\eta \geq 1$ such that for each $j \in \{0, 1, \dots, r-1\}$, $n \in \mathbb{N}$ and $x \in X_j$, if $n \equiv j \pmod{r}$ then*

$$p(n; x) = (2\pi n)^{-\frac{d}{2}} (\det B_s)^{-\frac{1}{2}} e^{-n\phi(\delta)} (r + \mathcal{O}(n^{-1} \text{dist}(\delta, \partial\mathcal{M})^{-2\eta})),$$

otherwise $p(n; x) = 0$, where $\delta = \frac{x}{n}$, $s = s(\delta)$ satisfies $\nabla \log \kappa(s) = \delta$, and

$$\phi(\delta) = \max \{ \langle u, \delta \rangle - \log \kappa(u) : u \in \mathbb{R}^d \}.$$

Proof. Using the Fourier inversion formula we can write

$$(20) \quad p(n; x) = \left(\frac{1}{2\pi} \right)^d \int_{\mathcal{D}_d} \kappa(i\theta)^n e^{-i\langle \theta, x \rangle} d\theta$$

where $\mathcal{D}_d = [-\pi, \pi)^d$. If $\theta_0 \in \mathcal{U}$ then $\kappa(i\theta_0) = e^{it}$ for some $t \in [-\pi, \pi)$ where \mathcal{U} is defined in (3). Since $\kappa(i\theta_0)$ is a convex combination of complex numbers from the unit circle, $\kappa(i\theta_0) = e^{it}$ if and only if $e^{i\langle \theta_0, v \rangle} = e^{it}$ for each $v \in \mathcal{V}$. In particular,

$$\begin{aligned} e^{int} p(n; x) &= \left(\frac{1}{2\pi} \right)^d \int_{\mathcal{D}_d} \kappa(i\theta + i\theta_0)^n e^{-i\langle \theta, x \rangle} d\theta \\ &= e^{i\langle \theta_0, x \rangle} p(n; x), \end{aligned}$$

thus, whenever $p(n; x) > 0$, we have

$$(21) \quad e^{int} = e^{i\langle \theta_0, x \rangle}.$$

Hence, by (5),

$$e^{int} = e^{i\langle \theta_0, x \rangle} = e^{i(n+r)t},$$

which implies that e^{it} is r 'th root of unity. In particular, the set \mathcal{U} has the cardinality r . Next, we claim that

Claim 2. *For any $u \in \mathbb{R}^d$,*

$$\int_{\mathcal{D}_d} \kappa(i\theta)^n e^{-i\langle \theta, x \rangle} d\theta = \int_{\mathcal{D}_d} \kappa(u + i\theta)^n e^{-\langle u + i\theta, x \rangle} d\theta.$$

To see this, we observe that for $y \in \mathbb{Z}^d$ we have

$$\begin{aligned} \int_{\mathcal{D}_d} e^{i\langle \theta, y \rangle} e^{-i\langle \theta, x \rangle} d\theta &= \prod_{j=1}^d \int_{-\pi}^{\pi} e^{i\theta_j y_j} e^{-i\theta_j x_j} d\theta_j \\ &= \prod_{j=1}^d \int_{-\pi}^{\pi} e^{(u_j + i\theta_j) y_j} e^{-(u_j + i\theta_j) x_j} d\theta_j \\ &= \int_{\mathcal{D}_d} e^{\langle u + i\theta, y \rangle} e^{-\langle u + i\theta, x \rangle} d\theta. \end{aligned}$$

Therefore,

$$\begin{aligned} \int_{\mathcal{D}_d} \kappa(i\theta)^n e^{-i\langle \theta, x \rangle} d\theta &= \sum_{v_1, \dots, v_n \in \mathcal{V}} \prod_{j=1}^n p(v_j) \int_{\mathcal{D}_d} e^{i\langle \theta, \sum_{j=1}^n v_j \rangle} e^{-i\langle \theta, x \rangle} d\theta \\ &= \sum_{v_1, \dots, v_n \in \mathcal{V}} \prod_{j=1}^n p(v_j) \int_{\mathcal{D}_d} e^{\langle u + i\theta, \sum_{j=1}^n v_j \rangle} e^{-\langle u + i\theta, x \rangle} d\theta \\ &= \int_{\mathcal{D}_d} \kappa(u + i\theta)^n e^{-\langle u + i\theta, x \rangle} d\theta. \end{aligned}$$

We notice that if $p(n; x) > 0$ then $\delta = \frac{x}{n} \in \overline{\mathcal{M}}$. Since $\text{dist}(\delta, \partial\mathcal{M}) > 0$, by Theorem 2.1, there is the unique $s = s(\delta)$ such that $\nabla \log \kappa(s) = \delta$. Hence, by Claim 2, we can write

$$p(n; x) = \left(\frac{1}{2\pi} \right)^d e^{-n\phi(\delta)} \int_{\mathcal{D}_d} \left(\frac{\kappa(s + i\theta)}{\kappa(s)} \right)^n e^{-i\langle \theta, x \rangle} d\theta.$$

Let $\epsilon > 0$ be small enough to satisfy (25), (27) and (30). We set

$$\mathcal{D}_d^\epsilon = \bigcap_{\theta_0 \in \mathcal{U}} \{ \theta \in [-\pi, \pi]^d : |\theta - \theta_0|_2 \geq \epsilon \}.$$

Then the integral over \mathcal{D}_d^ϵ is negligible. To see this, we write

$$\begin{aligned} 1 - \left| \frac{\kappa(s + i\theta)}{\kappa(s)} \right|^2 &= 1 - \sum_{v, v' \in \mathcal{V}} \frac{p(v) e^{\langle s + i\theta, v \rangle}}{\kappa(s)} \cdot \frac{p(v') e^{\langle s - i\theta, v' \rangle}}{\kappa(s)} \\ (22) \quad &= 2 \sum_{v, v' \in \mathcal{V}} \frac{p(v) e^{\langle s, v \rangle}}{\kappa(s)} \cdot \frac{p(v') e^{\langle s, v' \rangle}}{\kappa(s)} \left(\sin \left\langle \frac{\theta}{2}, v - v' \right\rangle \right)^2. \end{aligned}$$

Now, we need the following estimate.

Claim 3. *For every $v_0 \in \mathcal{V}$, there is $\xi > 0$ such that for all $\theta \in \mathcal{D}_d^\epsilon$ there is $v \in \mathcal{V}$ satisfying*

$$(23) \quad \left| \sin \left\langle \frac{\theta}{2}, v - v_0 \right\rangle \right| \geq \xi.$$

For the proof, we assume to contrary that for some $v_0 \in \mathcal{V}$ and all $m \in \mathbb{N}$ there is $\theta_m \in \mathcal{D}_d^\epsilon$ such that for all $v \in \mathcal{V}$

$$\left| \sin \left\langle \frac{\theta_m}{2}, v - v_0 \right\rangle \right| \leq \frac{1}{m}.$$

By compactness of \mathcal{D}_d^ϵ , there is a subsequence $(\theta_{m_k} : k \in \mathbb{N})$ convergent to $\theta' \in \mathcal{D}_d^\epsilon$. Then for all $v \in \mathcal{V}$

$$\sin \left\langle \frac{\theta'}{2}, v - v_0 \right\rangle = 0,$$

and hence

$$\kappa(i\theta') = e^{i\langle \theta', v_0 \rangle}$$

which is impossible since $\theta' \in \mathcal{D}_d^\epsilon$.

In order to apply Claim 3, we select any v_0 satisfying

$$\langle v_0, s \rangle = \max \{ \langle v, s \rangle : v \in \mathcal{V} \},$$

thus $e^{\langle s, v_0 \rangle} \geq \kappa(s)$. By Claim 3 and (22), for each $\theta \in \mathcal{D}_d^\epsilon$ there is $v \in \mathcal{V}$ such that

$$1 - \left| \frac{\kappa(s + i\theta)}{\kappa(s)} \right|^2 \geq 2p(v_0) \frac{p(v)e^{\langle s, v \rangle}}{\kappa(s)} \xi^2.$$

Although v may depend on θ , by Theorem 2.2, there are $C > 0$ and $\eta \geq 1$ such that for all $\theta \in \mathcal{D}_d^\epsilon$

$$1 - \left| \frac{\kappa(s + i\theta)}{\kappa(s)} \right|^2 \geq C \operatorname{dist}(\delta, \partial\mathcal{M})^\eta.$$

Hence,

$$\left| \frac{\kappa(s + i\theta)}{\kappa(s)} \right|^2 \leq 1 - C \operatorname{dist}(\delta, \partial\mathcal{M})^\eta \leq e^{-C \operatorname{dist}(\delta, \partial\mathcal{M})^\eta}.$$

Since

$$n \operatorname{dist}(\delta, \partial\mathcal{M})^\eta = n^{\frac{1}{2}} (n \operatorname{dist}(\delta, \partial\mathcal{M})^{2\eta})^{\frac{1}{2}}$$

we obtain that

$$e^{-Cn \operatorname{dist}(\delta, \partial\mathcal{M})^\eta} \leq C' n^{-\frac{d}{2}-1} \operatorname{dist}(\delta, \partial\mathcal{M})^{-2\eta},$$

provided n is large enough. Finally, since $\det B_s \leq 1$ we conclude that

$$\int_{\mathcal{D}_d^\epsilon} \left| \frac{\kappa(s + i\theta)}{\kappa(s)} \right|^n d\theta \leq C n^{-\frac{d}{2}-1} (\det B_s)^{-\frac{1}{2}} \operatorname{dist}(\delta, \partial\mathcal{M})^{-2\eta}.$$

Next, let us consider the integral over

$$(24) \quad \bigcup_{\theta_0 \in \mathcal{U}} \{ \theta \in [-\pi, \pi]^d : |\theta - \theta_0|_2 < \epsilon \}.$$

By taking ϵ satisfying

$$(25) \quad \epsilon < \min \left\{ \frac{|\theta_0 - \theta'_0|_2}{2} : \theta_0, \theta'_0 \in \mathcal{U} \right\},$$

we guarantee that the sets in (24) are disjoint. Moreover, for any $\theta_0 \in \mathcal{U}$, by the change of variables and (21) we get

$$\begin{aligned} \int_{|\theta - \theta_0|_2 < \epsilon} \left(\frac{\kappa(s + i\theta)}{\kappa(s)} \right)^n e^{-i\langle \theta, x \rangle} d\theta &= \int_{|\theta|_2 < \epsilon} \left(\frac{\kappa(s + i\theta + i\theta_0)}{\kappa(s)} \right)^n e^{-i\langle \theta + \theta_0, x \rangle} d\theta \\ &= \int_{|\theta|_2 < \epsilon} \left(\frac{\kappa(s + i\theta)}{\kappa(s)} \right)^n e^{-i\langle \theta, x \rangle} d\theta. \end{aligned}$$

Therefore,

$$\sum_{\theta_0 \in \mathcal{U}} \int_{|\theta - \theta_0|_2 < \epsilon} \left(\frac{\kappa(s + i\theta)}{\kappa(s)} \right)^n e^{-i\langle \theta, x \rangle} d\theta = r \int_{|\theta|_2 < \epsilon} \left(\frac{\kappa(s + i\theta)}{\kappa(s)} \right)^n e^{-i\langle \theta, x \rangle} d\theta.$$

Further, by (16), a function $z \mapsto \text{Log } \kappa(z)$, where Log denotes the principal value of the complex logarithm, is an analytic function in a strip $\mathbb{R}^d + iB$ where

$$B = \left\{ b \in \mathbb{R}^d : |b|_2 < (2 \cdot \max\{|v|_2 : v \in \mathcal{V}\})^{-1} \right\}.$$

Since for any $u \in \mathbb{R}^d$ we have

$$D_u^2 \text{Log } \kappa(z) = \frac{1}{2} \sum_{v, v' \in \mathcal{V}} \frac{p(v)e^{\langle z, v \rangle}}{\kappa(z)} \cdot \frac{p(v')e^{\langle z, v' \rangle}}{\kappa(z)} \langle u, v - v' \rangle^2,$$

by Lemma 2.4, there is $C > 0$ such that for all $\sigma \in \mathbb{N}^d$ and $a + ib \in \mathbb{R}^d + iB$

$$(26) \quad |\partial^\sigma (D_u^2 \text{Log } \kappa)(a + ib)| \leq C^{|\sigma|+1} \sigma! B_a(u, u).$$

If

$$(27) \quad \epsilon < (2 \cdot \max\{|v|_2 : v \in \mathcal{V}\})^{-1},$$

then for $|\theta|_2 < \epsilon$ we can define

$$\psi(s, \theta) = \text{Log } \kappa(s + i\theta) - \log \kappa(s) - i\langle \theta, \delta \rangle + \frac{1}{2} B_s(\theta, \theta).$$

Hence,

$$\int_{|\theta|_2 < \epsilon} \left(\frac{\kappa(s + i\theta)}{\kappa(s)} \right)^n e^{-i\langle \theta, x \rangle} d\theta = \int_{|\theta|_2 < \epsilon} e^{-\frac{n}{2} B_s(\theta, \theta)} e^{n\psi(s, \theta)} d\theta,$$

and to finish the proof of theorem it is enough to show

Claim 4.

$$\begin{aligned} (28) \quad \int_{|\theta|_2 < \epsilon} e^{-\frac{n}{2} B_s(\theta, \theta)} e^{n\psi(s, \theta)} d\theta \\ = (2\pi n)^{\frac{d}{2}} (\det B_s)^{-\frac{1}{2}} \left(1 + \mathcal{O}(n^{-1} \text{dist}(\delta, \partial \mathcal{M})^{-2\eta}) \right). \end{aligned}$$

Using the integral form for the reminder, we get

$$\psi(s, \theta) = -\frac{i}{2} \int_0^1 (1-t)^2 D_\theta^3 \text{Log } \kappa(s + it\theta) dt.$$

In view of (26), there is $C > 0$ such that for all $s \in \mathbb{R}^d$ and $\theta \in B$

$$(29) \quad |\psi(s, \theta)| \leq C|\theta|_2 B_s(\theta, \theta).$$

Therefore, by choosing

$$(30) \quad \epsilon < \left(4 \cdot \sup \left\{ \frac{|\psi(a, b)|}{|b|_2 B_a(b, b)} : a \in \mathbb{R}^d, b \in B \right\} \right)^{-1},$$

if $|\theta|_2 < \epsilon$ then we may estimate

$$(31) \quad |\psi(s, \theta)| \leq \frac{1}{4} B_s(\theta, \theta).$$

Next, we write

$$\begin{aligned} e^{n\psi(s, \theta)} &= \left(e^{n\psi(s, \theta)} - 1 - n\psi(s, \theta) \right) + \left(n\psi(s, \theta) - n \frac{D_\theta^3 \psi(s, 0)}{3!} \right) \\ &\quad + n \frac{D_\theta^3 \psi(s, 0)}{3!} + 1, \end{aligned}$$

and split (28) into four corresponding integrals.

Since for $a \in \mathbb{C}$

$$|e^a - 1 - a| \leq \frac{|a|^2}{2} e^{|a|},$$

by (29) and (31), the first integrand can be estimated as follows

$$\begin{aligned} \left| e^{n\psi(s, \theta)} - 1 - n\psi(s, \theta) \right| &\leq \frac{1}{2} e^{\frac{n}{4} B_s(\theta, \theta)} (n\psi(s, \theta))^2 \\ &\leq C e^{\frac{n}{4} B_s(\theta, \theta)} n^2 |\theta|_2^2 B_s(\theta, \theta)^2. \end{aligned}$$

Because

$$(32) \quad |\theta|_2^2 \leq \|B_s^{-1}\| B_s(\theta, \theta),$$

we obtain

$$\begin{aligned} (33) \quad &\left| \int_{|\theta|_2 < \epsilon} e^{-\frac{n}{2} B_s(\theta, \theta)} (e^{n\psi(s, \theta)} - 1 - n\psi(s, \theta)) \, d\theta \right| \\ &\leq C n^2 \|B_s^{-1}\| \int_{|\theta|_2 < \epsilon} e^{-\frac{n}{4} B_s(\theta, \theta)} B_s(\theta, \theta)^3 \, d\theta \\ &\leq C n^{-\frac{d}{2}-1} (\det B_s)^{-\frac{1}{2}} \|B_s^{-1}\|. \end{aligned}$$

Furthermore, by (26),

$$\begin{aligned} \left| \psi(s, \theta) - \frac{D_\theta^3 \psi(s, 0)}{3!} \right| &= \left| \frac{1}{3!} \int_0^1 (1-t)^3 D_\theta^4 \text{Log } \kappa(s + it\theta) \, dt \right| \\ &\leq C |\theta|_2^2 B_s(\theta, \theta), \end{aligned}$$

which together with (32) implies

$$\begin{aligned}
 & \left| \int_{|\theta|_2 < \epsilon} e^{-\frac{n}{2} B_s(\theta, \theta)} n \left(\psi(s, \theta) - \frac{D_\theta^3 \psi(s, 0)}{3!} \right) d\theta \right| \\
 & \leq Cn \|B_s^{-1}\| \int_{|\theta|_2 < \epsilon} e^{-\frac{n}{2} B_s(\theta, \theta)} B_s(\theta, \theta)^2 d\theta \\
 (34) \quad & \leq Cn^{-\frac{d}{2}-1} (\det B_s)^{-\frac{1}{2}} \|B_s^{-1}\|.
 \end{aligned}$$

The third integral is equal zero. The last one, by (32), we can estimate

$$\begin{aligned}
 & \left| \int_{|\theta|_2 < \epsilon} e^{-\frac{n}{2} B_s(\theta, \theta)} d\theta - \int_{\mathbb{R}^d} e^{-\frac{n}{2} B_s(\theta, \theta)} d\theta \right| \\
 & \leq e^{-\frac{n}{4} \epsilon^2} \|B_s^{-1}\|^{-1} \int_{\mathbb{R}^d} e^{-\frac{n}{4} B_s(\theta, \theta)} d\theta \\
 (35) \quad & \leq Cn^{-\frac{d}{2}-1} (\det B_s)^{-\frac{1}{2}} \|B_s^{-1}\|.
 \end{aligned}$$

By putting estimates (33), (34) and (35) together, we obtain

$$\int_{|\theta|_2 < \epsilon} e^{-\frac{n}{2} B_s(\theta, \theta)} e^{n\psi(s, \theta)} d\theta = n^{-\frac{d}{2}} (\det B_s)^{-\frac{1}{2}} \left((2\pi)^{\frac{d}{2}} + \mathcal{O}(n^{-1} \|B_s^{-1}\|) \right).$$

Finally, by (8) and Theorem 2.2, there is $C > 0$ such that for all $\delta \in \mathcal{M}$ and any $u \in \mathbb{R}^d$

$$(36) \quad B_0(u, u) \geq B_s(u, u) \geq C \operatorname{dist}(\delta, \partial\mathcal{M})^{2\eta} B_0(u, u).$$

Hence,

$$\|B_s^{-1}\| = \left(\min\{B_s(u, u) : |u|_2 = 1\} \right)^{-1} \leq C \operatorname{dist}(\delta, \partial\mathcal{M})^{-2\eta},$$

which concludes the proof of Claim 4. \square

Although, the asymptotic in Theorem 3.1 is uniform on a large region with respect to n and x , it depends on the implicit function $s(\delta)$. By (36), we may estimate

$$1 \geq \det B_s \geq C \operatorname{dist}(\delta, \partial\mathcal{M})^{2r\eta}.$$

Since $\mathcal{M} \ni \delta \mapsto s(\delta)$ is real analytic, for each $\epsilon > 0$ there is $C_\epsilon > 0$ such that if $\operatorname{dist}(\delta, \partial\mathcal{M}) \geq \epsilon$ then

$$(37) \quad |s|_1 \leq C_\epsilon |\delta - \delta_0|_1,$$

and

$$\left| (\det B_s)^{-\frac{1}{2}} - (\det B_0)^{-\frac{1}{2}} \right| \leq C_\epsilon |\delta - \delta_0|_1$$

where $\delta_0 = \sum_{v \in \mathcal{V}} p(v)v$. In the most applications the following form of the asymptotic of $p(n; x)$ is sufficient.

Corollary 3.2. *For every $\epsilon > 0$, $j \in \{0, \dots, r-1\}$, $n \in \mathbb{N}$ and $x \in X_j$, if $n \equiv j \pmod{r}$ then*

$$p(n; x) = (2\pi n)^{-\frac{d}{2}} (\det B_0)^{-\frac{1}{2}} e^{-n\phi(\delta)} \left(r + \mathcal{O}(|\delta - \delta_0|_1) + \mathcal{O}(n^{-1}) \right),$$

otherwise $p(n; x) = 0$, provided that $\text{dist}(\delta, \partial\mathcal{M}) \geq \epsilon$.

Remark 1. It is not possible to replace $\phi(\delta)$ by $\frac{1}{2}B_0^{-1}(\delta - \delta_0, \delta - \delta_0)$ without introducing an error term of a very different nature. Indeed, by (12),

$$e^{-n\phi(\delta)} = e^{-\frac{n}{2}B_0^{-1}(\delta - \delta_0, \delta - \delta_0)} e^{\mathcal{O}(n|\delta - \delta_0|^3)}.$$

Since $\delta_0 \in \mathcal{M}$, if δ approaches $\partial\mathcal{M}$ then $n|\delta - \delta_0|^3$ cannot be small. Notice that the third power may be replaced by an higher degree if the random walk has vanishing moments. In particular, for the simple random walk on \mathbb{Z}^d (see Example 1), for all $\epsilon > 0$, $x \in \mathbb{Z}^d$ and $n \in \mathbb{N}$, if $|x|_1 + n \in 2\mathbb{N}$ then

$$p(n; x) = (2\pi)^{-\frac{d}{2}} \left(\frac{d}{n}\right)^{\frac{d}{2}} e^{-\frac{d}{2n}|x|_2^2} (2 + \mathcal{O}(n|\delta|^4) + \mathcal{O}(n^{-1}))$$

otherwise $p(n; x) = 0$, uniformly with respect to n and x provided that $|x|_1 \leq (1 - \epsilon)n$.

Remark 2. It is relatively easy to obtain a global upper bound: for all $n \in \mathbb{N}$ and $x \in \mathbb{Z}^d$

$$p(n; x) \leq e^{-n\phi(\delta)}.$$

Indeed, by Claim 2, for $u \in \mathbb{R}^d$, we have

$$p(n; x) = \left(\frac{1}{2\pi}\right)^d \kappa(u)^n e^{-\langle u, x \rangle} \int_{\mathcal{D}_d} \left(\frac{\kappa(u + i\theta)}{\kappa(u)}\right)^n e^{-i\langle \theta, x \rangle} d\theta.$$

Hence, by Theorem 2.1,

$$p(n; x) \leq \min \{ \kappa(u)^n e^{-\langle u, x \rangle} : u \in \mathbb{R}^d \} \leq e^{-n\phi(\delta)}.$$

APPENDIX A. APPLICATIONS

In this section we apply Corollary 3.2 to simple random walks on triangular and hexagonal lattices.

A.1. The triangular lattice. The triangular lattice consists of the set of points

$$L = \mathbb{Z}\lambda_1 \oplus \mathbb{Z}\lambda_2,$$

where $\lambda_1 = (-1/2, \sqrt{3}/2)$, $\lambda_2 = (1/2, \sqrt{3}/2)$. Let $\tau : L \rightarrow \{0, 1, 2\}$ be defined by setting

$$\tau(j\lambda_1 + j'\lambda_2) \equiv j + 2j' \pmod{3}.$$

Each point $x \in L$ has six closest neighbours, namely,

$$\begin{aligned} x + \lambda_1, \quad x - \lambda_1, \quad x + \lambda_1 - \lambda_2, \\ x + \lambda_2, \quad x - \lambda_2, \quad x - \lambda_1 + \lambda_2. \end{aligned}$$

Let p be the transition function of the simple random walk on L . Observe that the mapping

$$L \ni j\lambda_1 + j'\lambda_2 \mapsto je_1 + j'e_2 \in \mathbb{Z}^2$$

allows us instead of p to work with a transition function \tilde{p} such that

$$\tilde{p}(e_1) = \tilde{p}(-e_1) = \tilde{p}(e_1 - e_2) = \tilde{p}(e_2) = \tilde{p}(-e_2) = \tilde{p}(-e_1 + e_2) = \frac{1}{6}.$$

Then the corresponding set $\mathcal{M} \subset \mathbb{R}^2$ is the interior of

$$\text{conv} \{e_1, e_2, e_1 - e_2, -e_1, -e_2, -e_1 + e_2\}.$$

Moreover, for $u \in \mathbb{R}^2$

$$\begin{aligned} \kappa(u) &= \frac{1}{6}e^{u_1} + \frac{1}{6}e^{-u_1} + \frac{1}{6}e^{u_2} + \frac{1}{6}e^{-u_2} + \frac{1}{6}e^{u_1-u_2} + \frac{1}{6}e^{-u_1+u_2} \\ &= \frac{1}{3} \cosh u_1 + \frac{1}{3} \cosh u_2 + \frac{1}{3} \cosh(u_1 - u_2). \end{aligned}$$

In particular, $\mathcal{U} = \{0\}$. Next, by using (8) we can calculate the quadratic form B_0 ,

$$B_0 = \frac{1}{3} \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix},$$

thus $\det B_0 = 1/3$. Finally, by applying Corollary 3.2 to \tilde{p} we obtain the following precise asymptotic of p : for every $\epsilon > 0$ and all $n \in \mathbb{N}$ and $x \in L$

$$(38) \quad p(n; x) = (2\pi n)^{-1} e^{-n\phi(\delta)} (\sqrt{3} + \mathcal{O}(|\delta|_1) + \mathcal{O}(n^{-1})),$$

uniformly with respect to n and x such that $\text{dist}(\delta, \partial\mathcal{M}) \geq \epsilon$ where for $x = j\lambda_1 + j'\lambda_2$ we have set

$$\delta = \frac{j}{n}e_1 + \frac{j'}{n}e_2.$$

The reader may wish to compare the asymptotic (38) with known results (see [5, Example 2], [6] and [4, Section 7.2]).

A.2. The hexagonal lattice. The hexagonal lattice H one may obtain from the triangular lattice by removing all vertices $x \in L$ such that $\tau(x) = 1$. Each vertex $x \in H$ has three neighbours,

$$\begin{cases} x + \lambda_2, & x - \lambda_1, & x - \lambda_2 + \lambda_1, & \text{if } \tau(x) = 0 \\ x + \lambda_1, & x - \lambda_2, & x - \lambda_1 + \lambda_2, & \text{if } \tau(x) = 2. \end{cases}$$

Let p be the transition function of the simple random walk on H , i.e. $p(x, y) = 1/3$ if x and y are closest neighbours. Observe that p is irreducible and periodic with period $r = 2$. We have

$$X_0 = \{x \in H : \tau(x) = 0\}, \quad X_1 = \{x \in H : \tau(x) = 2\}.$$

Let us consider a new random walk given by a transition function q

$$q(x, y) = \sum_{\substack{u \sim x \\ u \sim y}} p(x, u)p(u, y),$$

where the sum is taken over $u \in H$ being a common neighbour of x and y . It is easy to check that $q(x, x) = 1/3$ and $q(x, y) = 1/9$ where y belongs to the set

$$\begin{aligned} &x + \lambda_1 + \lambda_2, \quad x + 2\lambda_1 - \lambda_2, \quad x + \lambda_1 - 2\lambda_2, \\ &x - \lambda_1 - \lambda_2, \quad x - \lambda_1 + 2\lambda_2, \quad x - 2\lambda_1 + \lambda_2. \end{aligned}$$

If $x \in X_0$ then

$$p(n; 0, x) = \begin{cases} q(n/2; 0, x) & \text{if } n \equiv 0 \pmod{2}, \\ 0 & \text{otherwise.} \end{cases}$$

If $x \in X_1$ and $n \equiv 1 \pmod{2}$ then

$$\begin{aligned} p(n; 0, x) &= \frac{1}{3}q((n-1)/2; 0, x + \lambda_1) \\ &\quad + \frac{1}{3}q((n-1)/2; 0, x - \lambda_2) \\ &\quad + \frac{1}{3}q((n-1)/2; 0, x + \lambda_1 - \lambda_2) \end{aligned}$$

otherwise $p(n; 0, x) = 0$. First, we find the asymptotic of q . We notice that q is the transition function of an irreducible random walk on the triangular lattice

$$L = \mathbb{Z}(\lambda_1 + \lambda_2) \oplus \mathbb{Z}(2\lambda_2 - \lambda_1).$$

Therefore, under the mapping

$$L \ni j\lambda_1 + j'\lambda_2 \mapsto \frac{2j+j'}{3}e_1 + \frac{-j+j'}{3}e_2 \in \mathbb{Z}^2$$

the transition function q is mapped onto \tilde{q} a transition function of a random walk on the integer lattice \mathbb{Z}^2 where

$$\begin{aligned} \tilde{q}(0) &= \frac{1}{3}, \\ \tilde{q}(e_1) &= \tilde{q}(-e_1) = \tilde{q}(e_2) = \tilde{q}(-e_2) = \tilde{q}(e_1 - e_2) = \tilde{q}(e_2 - e_1) = \frac{1}{9}. \end{aligned}$$

In this case, for $u \in \mathbb{R}^2$ we have

$$\begin{aligned} \kappa(u) &= \frac{1}{3} + \frac{1}{9}e^{u_1} + \frac{1}{9}e^{-u_1} + \frac{1}{9}e^{u_2} + \frac{1}{9}e^{-u_2} + \frac{1}{9}e^{u_1-u_2} + \frac{1}{9}e^{-u_1+u_2} \\ &= \frac{1}{3} + \frac{2}{9}\cosh u_1 + \frac{2}{9}\cosh u_2 + \frac{2}{9}\cosh(u_1 - u_2). \end{aligned}$$

Again, the set $\mathcal{M} \subset \mathbb{R}^2$ is the interior of

$$\text{conv}\{e_1, e_2, e_1 - e_2, -e_1, -e_2, -e_1 + e_2\}.$$

It is easy to calculate that the quadratic form B_0 is equal to

$$B_0 = \frac{2}{9} \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix},$$

thus $\det B_0 = 4/27$. By Corollary 3.2, for every $\epsilon > 0$, all $n \in \mathbb{N}$ and $x \in L$

$$(39) \quad q(n; x) = (2\pi n)^{-1} e^{-n\phi(\delta)} (3\sqrt{3} + \mathcal{O}(|\delta|_1) + \mathcal{O}(n^{-1}))$$

uniformly with respect to n and x such that $\text{dist}(\delta, \partial\mathcal{M}) \geq \epsilon$ where for $x = j\lambda_1 + j'\lambda_2$ we have set

$$(40) \quad \delta = \frac{2j+j'}{3n}e_1 + \frac{-j+j'}{3n}e_2.$$

Although, for $x \in X_1$, we need to apply (39) for three times with different x , the exponential factors are comparable. Indeed, for $x \in H$, $x = j\lambda_1 + j'\lambda_2$ let δ be defined by the formula (40). Fix $\epsilon > 0$ and let us consider $x \in X_1$ and $n \in \mathbb{N}$ such that

$$\text{dist}(\delta, \partial\mathcal{M}) \geq \epsilon.$$

Let $\tilde{\delta}$ be any element of a set

$$\delta + \left\{ \frac{2}{3n}e_1 - \frac{1}{3n}e_2, -\frac{1}{3n}e_1 - \frac{1}{3n}e_2, \frac{1}{3n}e_1 - \frac{2}{3n}e_n \right\}.$$

Observe that

$$(41) \quad |\delta - \tilde{\delta}|_1 \leq \frac{1}{n}.$$

Let us denote by $s \in \mathbb{R}^2$ the unique solution to $\nabla \log \kappa(s) = \delta$, then by (10), we can estimate

$$\phi(\delta) - \phi(\tilde{\delta}) \leq \phi(\delta) - \langle \tilde{\delta}, s \rangle + \log \kappa(s) = \langle \delta - \tilde{\delta}, s \rangle.$$

Hence, by (37) and (41), we get

$$\phi(\delta) - \phi(\tilde{\delta}) \leq C_\epsilon \frac{1}{n} |\delta|_1.$$

In particular,

$$e^{-n\phi(\tilde{\delta})} = e^{-n\phi(\delta)} e^{n(\phi(\delta) - \phi(\tilde{\delta}))} = e^{-n\phi(\delta)} (1 + \mathcal{O}(|\delta|_1)).$$

Now, we are ready to apply (39) to obtain the precise asymptotic of p . For every $\epsilon > 0$ and all $x \in H$, $n \in \mathbb{N}$, $j \in \{0, 1\}$, if $x \in X_j$ then

$$p(n; 0, x) = \begin{cases} (2\pi n)^{-1} e^{-n\phi(\delta)} (3\sqrt{3} + \mathcal{O}(|\delta|_1) + \mathcal{O}(n^{-1})), & \text{if } n \equiv j \pmod{2}, \\ 0, & \text{otherwise,} \end{cases}$$

uniformly with respect to n and x such that $\text{dist}(\delta, \partial\mathcal{M}) \geq \epsilon$. The reader may compare the above asymptotic with [4, Section 7.3], [5, Example 3] and [6].

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